The Morrison Cone Conjecture under deformation

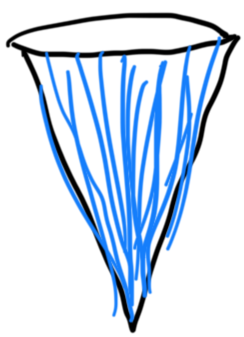
arxiv:2410.05949

Y Calabi-Yau 3fold, i.e.

Y smooth proj. variety, $K_Y = 0$, $h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = 0$.

Notation: if L f.g free ab. gp., $C \subset L_{\mathbb{R}}$ cone,
define $C^+ = \mathrm{Conv}(\overline{C} \cap L)$ "rational closure"

e.g. $C =$ positive cone of K3



$C^+ = C \cup$ rational rays.

__Conjecture__ (Morrison '93)

1) $\mathrm{Aut}\, Y \curvearrowright \mathrm{Nef}^+ Y$ w/ RPFD
2) $\mathrm{Bir}\, Y \curvearrowright \underline{\mathrm{Mov}}^+ Y$ w/ RPFD (Kawamata '97)

↳ movable divisors

In particular, finitely many orbits on faces

Remarks/Background:

1) MCC sometimes stated using $\mathrm{Nef}^+Y := \mathrm{Nef}\,Y \cap \ldots$ instead.

[Generalized Abundance Conjecture: D nef on CY3 Y $\Rightarrow$ D semiample — this would imply

$$\mathrm{Nef}^+Y = \mathrm{Nef}^e Y.$$]

2) Any birational automorphism of Y is a composite of flops.

D movable, $\exists\, \alpha : Y \overset{\text{flops}}{\dashrightarrow} Y'$ s.t $\alpha_* D$ nef

(run $(K_Y + \varepsilon D)$-MMP $\quad (Y, \varepsilon D)$ klt.

$\underset{\varepsilon D}{\overset{\shortparallel}{}}$

$$\therefore\ \mathrm{Mov}^e Y = \bigcup_{\substack{\alpha : Y \dashrightarrow Y' \\ \text{flops}}} \alpha^* \mathrm{Nef}^e Y'$$

3) MCC(2) $\Rightarrow$ $\mathrm{Bir}\,Y \curvearrowright \mathrm{Mov}\,Y$ w/ finitely many orbits on chambers $\mathrm{Nef}\,Y'$ for $Y \dashrightarrow Y'$ flops
$\Leftrightarrow$ finiteness of minimal models up to iso.

4) MCC only known in special cases

[...]

- generic Schoen CY3 (Grassi-Morrison)

[$X, X' \to \mathbb{P}^1$ generic rat. elliptic surfaces, discriminant loci disjoint $\quad Y = X \times_{\mathbb{P}^1} X'$ CY3]

- generic $(2,2,2,2) \subset (\mathbb{P}^1)^4$ (Cantat-Oguiso)
- Horrocks-Mumford quintic (Borcea-Fryers)

5) MCC generalized by Totaro to klt log CY pairs (Y, D).

<u>Theorem</u>: (L '24) If MCC holds for a CY3 Y, it holds for any CY3 deformation-equivalent to Y.

Enough to show $MCC(Y) \Leftrightarrow MCC(Y^{gen})$ where Y^{gen} is a very general deformation of Y.

(Wilson '92)

(1) Relate Nef Y & Nef Y^{gen}

Mov Y & Mov Y^{gen}

(2) Relate Aut Y & Aut Y^{gen}

Bir Y & Bir Y^{gen}

$h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = 0.$

(1) — $\mathrm{Nef}\, Y \subset \mathrm{Nef}\, Y^{gen}$ $\left(\mathrm{Pic}\, Y \cong H^2(Y) \cong H^2(Y^{gen})\right)$

— $\mathrm{Nef}\, Y \cap \mathrm{Big}\, Y$ locally rat. polyhedral (cone thm)

— faces $F \Leftrightarrow \pi_F : Y \to \overline{Y}$ birational contractions

$$\pi_F(C) = pt \Leftrightarrow [C] \in F^{\perp}$$

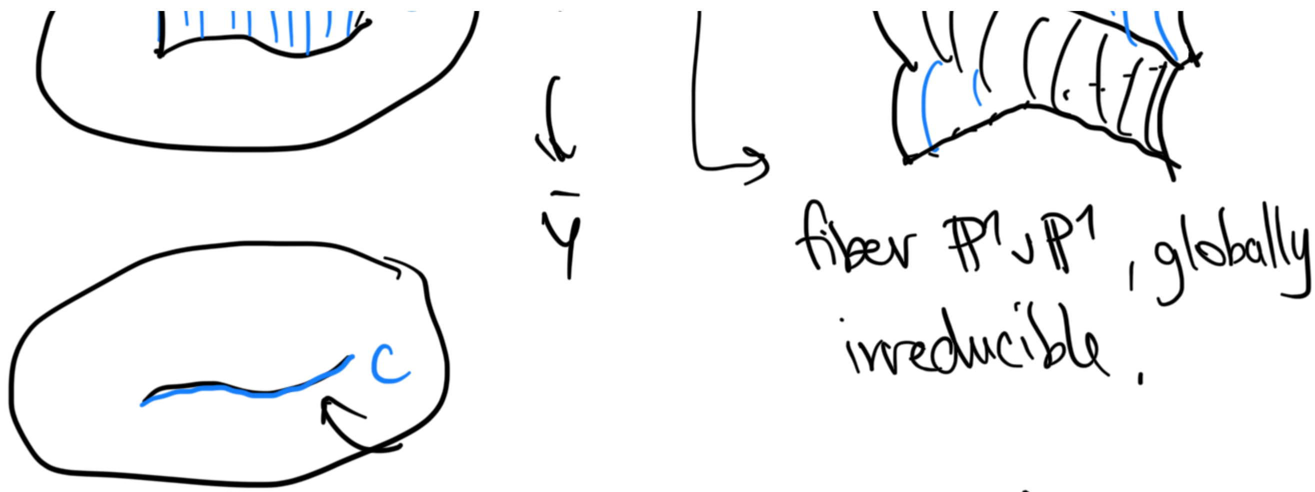
If $\mathrm{codim} F = 1$, all such $[C]$ lie in same ray R_F of $\overline{NE}(Y)$.

codim 1 face F disappears under deformation $\Leftrightarrow$

R_F contains no effective curves on Y^{gen}.

Wilson: this happens $\Leftrightarrow$ π_F :
$$\begin{array}{ccc} Y & \to & \overline{Y} \\ \cup & & \cup \\ E & \to & C \end{array}$$

E $\underline{\text{(quasi)}}$-ruled surface, $g(C) = 1$



In fact, if $E \subset Y$ ruled over a curve of genus $g > 0$.
E doesn't deform, $2g-2$ fibers do.

Similarly, - $\overline{\mathrm{Mov}}\,Y \subset \overline{\mathrm{Mov}}\,Y^{gen}$

- $\overline{\mathrm{Mov}}\,Y \cap \mathrm{Big}(Y)$ loc. rat. polyh.

- faces $F \iff Y \overset{\text{flops}}{\dashrightarrow} Y' \xrightarrow{\pi_F} \bar{Y}$

- F disappears $\iff$

$$
\begin{array}{ccc}
\pi_F : Y' & \longrightarrow & \bar{Y} \\
\cup & & \cup \\
E & \longrightarrow & C
\end{array}
$$

E birationally quasi-ruled, $g(C) \geq 1$

Define $\Delta_Y^{big} = \{ (E, \ell) \in H^2(Y) \times H_2(Y) \mid$ E birationally quasi-ruled over C, $g(C) \geq 1$, ℓ = class of fiber $\}$.

$$\Delta_Y^{sm} = \left\{ (E,\ell) \in H^2(Y) \times H_2(Y) \;\middle|\; \begin{array}{l} E \text{ quasi-ruled over} \\ C,\ g(C) = 1 \end{array} \right\}.$$

$$\therefore\ \mathrm{Nef}\,Y = \mathrm{Nef}\,Y^{gen} \cap \underline{\bigcap_{(E,\ell) \in \Delta_Y^{sm}}} \ell \geq 0$$

"fundamental chamber"

sim for Mov.

$E \cdot \ell = -2$ by adjunction formula

$\therefore\ \Delta_Y^{kij}$ "generalized simple roots"

$$\sigma_E : H^2(Y) \longrightarrow H^2(Y)$$
$$x \longmapsto x + \underline{(x,\ell)}E$$

$\sigma_E^2 = \mathrm{id}$

(Wilson 90s)

$$W_Y^{sm} = \langle \sigma_E \mid (E,\ell) \in \Delta_Y^{sm} \rangle$$
$$W_Y^{kij} = \langle \sigma_E \mid (E,\ell) \in \Delta_Y^{kij} \rangle$$

$m_{EE'}$	$(E,\ell')(E',\ell)$
1	$E = E'$
2	0
3	1
4	2
6	3
∞	o/w

Coxeter groups (only relation $(\sigma_E \sigma_{E'})^{\underline{m_{EE'}}} = 1$)

$W_Y^{sm} \subset$ Monodromy (c.f "fibered Dehn twist")

E.g. $f : Y \to \bar{Y}$ contraction to curve C of

Ex: -

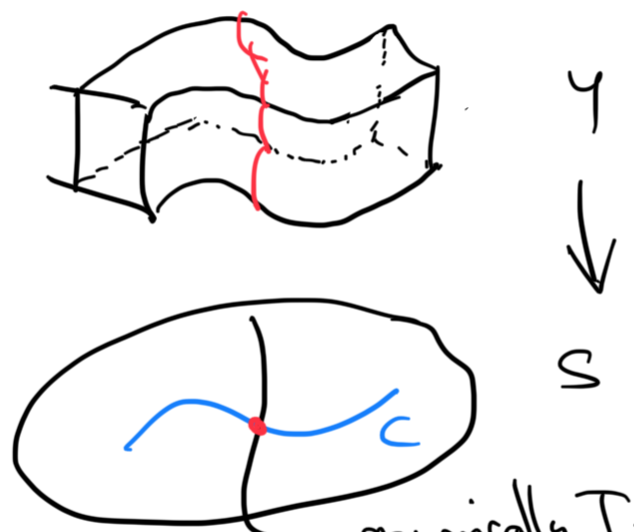
cDV, $g(C) > 0$.

$W_Y^{big} \supset$ Weyl group of corresponding du Val.

- $f : Y \to S$ elliptic CY3.

Suppose $C \subset \Delta$ comp. of discriminant locus of genus > 0. $f^{-1}(p)$ Kodaira fibre for $p \in C$ generic.

$W_Y^{big} \supset$ Weyl group of associated (folded) affine Dynkin diagram (cf. Szendrői)



generically I_4.

proved by Wilson in the case that $W_Y^{sm} < \infty$

Thm: (Wilson '90s, L '24)

$W_Y^{sm} \curvearrowright \mathrm{Nef}\,Y^{gen} \cap \mathrm{Big}(Y^{gen})$ w/ FD $\mathrm{Nef}\,Y \cap \mathrm{Big}(Y)$

$W_Y^{big} \curvearrowright \mathrm{Mov}\,Y^{gen} \cap \mathrm{Big}(Y^{gen})$ w/ FD $\mathrm{Mov}\,Y \cap \mathrm{Big}(Y)$.

<u>Proof</u>: – Generalize Bourbaki
(various E not lin. indep in $H^2(Y)$).

– Given $D \in \mathrm{Nef}\,Y^{gen} \cap \mathrm{Big}(Y^{gen})$, run relative MMP for $\underset{\varepsilon D}{\overset{\shortparallel}{K_Y+\varepsilon D}}$ over $S = \mathrm{Def}\,Y$.

every step in MMP is

$$\begin{tikzcd}
Y \arrow[d] & \subset \mathcal{Y} \arrow[dr] \arrow[rr, dashed, "\text{flop } \ell"] & & \mathcal{Y}' \arrow[dl] & \cong \mathcal{Y} \arrow[d] & \supset Y \arrow[d] \\
0 \in & & \mathrm{Def}\,Y \arrow[rr, "\sim"] \arrow[rr, bend right, "\text{universality } H^0(T_Y)=0"'] & & \mathrm{Def}\,Y & \ni 0
\end{tikzcd}$$

Composite $H^2(Y) \cong H^2(\mathcal{Y}) \cong H^2(\mathcal{Y}') \cong H^2(Y)$ is σ_E.
(cf. K3 surfaces)

$\rightsquigarrow w(D) \in \mathrm{Nef}\,Y$, some $w \in W_Y^{sm}$.

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(2) Relate $\mathrm{Aut}\,Y$ & $\mathrm{Aut}\,Y^{gen}$

Kollár $Y \dashrightarrow Y'$ flop of

Let $\Gamma = \mathrm{Aut}\,Y \ltimes W_Y^{sm}$. ($Y_s$ induces Hodge isometry $H^3(Y) \xrightarrow{\sim} H^3(Y')$)

$\Gamma \curvearrowright H^3(Y,\mathbb{Z})$ via Hodge isometries

$\Gamma \curvearrowright \mathrm{Def}\,Y$. Compatible with period map

$$P : \mathrm{Def}(Y) \to \mathbb{P} H^3(Y,\mathbb{C}) \quad \text{(immersion by}$$
$$s \to [\Omega_{Y_s}] \quad \text{inf. Torelli)}$$

<u>Theorem</u>: $\Gamma \curvearrowright \mathrm{Def}\,Y$ via finite group.

<u>Proof</u>: $h^1(\mathcal{O}_Y) = 0 \Rightarrow H^3(Y,\mathbb{C})$ primitive, so carries pos. def. Hermitian form, preserves unitary group.

Γ also preserves $H^3(Y,\mathbb{Z})$

$\Rightarrow \Gamma$ = compact + discrete $\Rightarrow$ finite.

Similarly $\mathrm{Bir}\,Y \ltimes W_Y^{bir} \curvearrowright \mathrm{Def}\,Y$ via finite group (cf. Kollár).

Let $K = \ker(\Gamma \curvearrowright \mathrm{Def}\,Y)$. $K \subset \Gamma$ finite index

<u>Proof of</u> $MCC(Y) \Leftrightarrow MCC(Y^{gen})$

$\mathrm{Aut}\,Y \circlearrowright \mathrm{Nef}^+ Y$ w/ RPFD (cf. Looijenga)

$\iff \Gamma = \mathrm{Aut}\,Y \ltimes W_Y^{sm} \circlearrowright \mathrm{Nef}^+ Y^{gen}$ w/ RPFD (Thm 1)

$\iff K \circlearrowright \mathrm{Nef}^+ Y^{gen}$ w/ RPFD (Looijenga)

$\iff \mathrm{Aut}\,Y^{gen} \circlearrowright \mathrm{Nef}^+ Y^{gen}$ w/ RPFD (work!)

$\Rightarrow$ easy ($K \subset \mathrm{Aut}(Y^{gen})$)

$\Leftarrow$ NTS that $\underline{\mathrm{Aut}\,Y^{gen} \subset \mathrm{Aut}\,Y \ltimes W_Y^{sm}} = \Gamma$

uses that $\mathrm{Aut}\,Y^{gen}$ f.g. by MCC.

Use results of Looijenga throughout.

Application: The work of Cantat-Oguiso

Y^{gen} = generic $(2,2,2,2) \subset (\mathbb{P}^1)^4$ (they also do $n > 4$.)

CY hypersurface $\xRightarrow{\text{Kollár}}$ Nef Y rat. polyh.

Consider Y^{gen} $Y^{gen} = \{F_1 x_0^2 + F_{11} x_0 x_1 + F_{12} x_1^2 = 0\}$

Consider Y $\quad Y = \{ \ldots \}$

$$\begin{array}{c} Y \\ \downarrow \pi_i \\ (\mathbb{P}^1)^3 \end{array}$$

π_i generically 2-1.

Over $B_i = \{F_{i,1} = F_{i,2}, F_{i,3} = 0\}$ fiber is a $\mathbb{P}^1$.

Since Y^{gen} generic, $B_i = 48$ pts.

$i_i : Y^{gen} \dashrightarrow Y^{gen}$ involution, flops these 48 $\mathbb{P}^1$'s.

Cantat-Oguiso: $\mathrm{Bir}\, Y^{gen} = \langle i_1, \ldots, i_4 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$

- $\mathrm{Nef}\, Y^{gen}$ is fund. domain for $\mathrm{Bir}\, Y^{gen} \curvearrowright \mathrm{Mov}\, Y^{gen}$

$\Rightarrow$ Movable cone conjecture holds for Y^{gen}.

Suppose Y smooth, but $F_{i,1}, F_{i,2}, F_{i,3}$ lin dependent.

Then $B_i = \{F_{i,1} = F_{i,2}, F_{i,3} = 0\}$ curve of genus $g = 25$.

$Y \rightsquigarrow Y^{gen}$ $2g-2=48$ fibers of ruling deform.

$\exists$ degeneration Y where $E_1,\dots,E_4$ effective.

$$W_Y^{bij} = \langle \sigma_{E_i} \mid i=1,\dots,4 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$$

$(E_i \cdot \ell_j) = 2$ if $i \neq j$

so no relations.

$$\mathrm{Mov}\,Y = \mathrm{Nef}\,Y = \mathrm{Nef}\,Y^{gen}.$$

So $W_Y^{bij} \curvearrowright \mathrm{Mov}\,Y^{gen}$ with FD $\mathrm{Mov}\,Y = \mathrm{Nef}\,Y^{gen}$.

$\Rightarrow$ recover Thm of Cantat–Oguiso (in dim 3).

$\Longrightarrow$ MCC holds for any smooth $(2,2,2,2) \subset (\mathbb{P}^1)^4$

Z smooth proj K3 w/ antisymplectic involution ι

E elliptic curve, involution $i: E \to E$.

$$\bar{Y} = Z \times E / (\iota, i) \quad \text{canonical CY3.}$$

$Y \to \bar{Y}$ crepant resolution (cf Borcea–Voisin)

<u>Thm</u>: If Z branched cover of $\mathbb{P}^2$ along generic sextic.. Then MCC holds for Y & any deformation. $\underline{h^{2,1}(Y) = 60}$.

<u>Proof</u>: Use MCC for KH log CY3 (Xu '24) to get MCC for Y.